Analysis of Finite-Precision Adaptive Filters Part I: Computation of the Residual Signal Variance

Analyse wertdiskreter Adaptionsverfahren
Teil I: Berechnung der Restsignal-Varianz

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Abstract:

Adaptive filters with variable coefficients have found many applications. The stochastic gradient least mean squares (LMS) algorithm is the most attractive adaptation scheme because of its computional simplicity. A fully digital implementation implies finite-precision adaptation algorithms with limited wordlength. For this condition exact theoretical investigations are currently available in literature only in part.

This contribution presents new results for finite-precision adaptive filters. Part I investigates convergence in terms of the residual signal variance caused by coefficient dithering. In Part II [2], the probability density function of the residual signal is computed which allows the evaluation of error probabilities in digital receivers.

Übersicht:

Für adaptive Filter mit variablen Koeffizienten gibt es viele Anwendungen. Das stochastische Gradientenverfahren (LMS-Algorithmus) ist aufgrund der einfachen Rechenoperationen das attraktivste Adaptionsverfahren. Eine volldigitale Realisierung führt zu wertdiskreten Adaptionsalgorithmen mit beschränkten Wortlängen, deren exakte theoretische Untersuchung bisher nur in Ansätzen erfolgt ist.

Es werden neue Ergebnisse zu wertdiskreten adaptiven Filtern präsentiert. In Teil I wird das Konvergenzverhalten anhand der Varianz eines immer vorhandenen Restsignals untersucht. In Teil II [2] wird die Verteilungsdichtefunktion des Restsignals erfaßt, was die Berechnung von Fehlerwahrscheinlichkeiten in digitalen Empfängern ermöglicht.

Für die Dokumentation:

Adaptive Filter / Echokompensation / LMS-Algorithmus / Koeffizientenzittern / Quantisierung / Restfehler / Konvergenzgeschwindigkeit

1. Introduction

The principal mode of operation of adaptive filters is not very difficult [3, 4] if the well-known LMS algorithm is used and any other conditions are supposed to be ideal. The investigations into this topic are almost completed with the fundamental paper by Mazo [5].

The theoretical analysis of finite-precision adaptation schemes is more difficult than the infinite-precision mode. The well-known theory of finite-precision filters with fixed coefficients is insignificant for adaptive filters – for instance the representation of coefficients in adaptive filters requires a much longer wordlength than in fixed filters. For a fast and reliable design, a comprehensive theory should be worked out even for finite-precision operation. But only a few references mostly restricted to particular aspects are available [6–13]. The most accurate results for sign adaptation can be found in [6, 7] and these contributions have served as starting point for the methods presented in this contribution.

An echo canceller as common used for full-duplex data transmission is introduced in section 2 as a typical example for finite impulse response (FIR) compensation filters. There is a close similarity to other adaptive filtering applications, e.g. decision feedback equalization. Section 3 describes the adaptation model and algorithm. Section 4 contains the theoretical analysis of finite-precision adaptation with emphasis on algorithms easy to implement, speed of convergence, and analysis of steady-state error. Finally in section 5, some simulation examples demonstrate the validity of the theoretical results.

The following abbreviations are used for least mean squares adaptation algorithms: C-LMS for infinite-precision operation (continuous mode), Q-LMS for finite-precision operation (quantized mode) and S-LMS for sign adaptation as a special case of Q-LMS.

2. Echo canceller as an example of compensation filter

The echo canceller configuration to be considered is sketched in **Fig. 1.** A discussion of this full-duplex data transmission system can be found in [8]. The known data sequence a_k is to be transmitted to the far-end transceiver at the other side of the communication channel. This far-end transceiver transmits simultaneously another, unknown data sequence which appears as the received far-end signal u_k. Due to certain imperfections, the signal u_k is corrupted by an echo signal y_k. It is assumed that the

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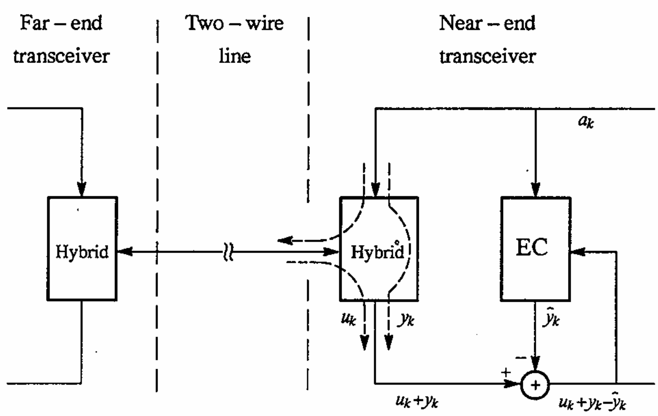

Fig. 1: Echo canceller configuration in a full-duplex communication system

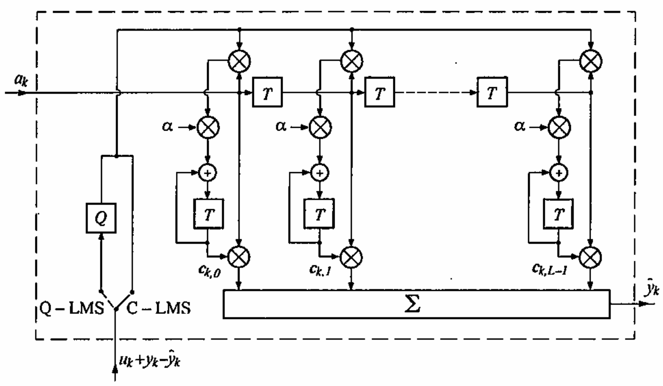

Fig. 2: Adaptive filter

unknown echo path can be modeled as linear filter with input a_k and output y_k.

The echo canceller has to produce an echo replica $\hat{y}_k$ for the unknown echo y_k. The residual echo $y_k - \hat{y}_k$ should be as small as possible in order to obtain a less corrupted signal u_k. Consequently the echo canceller has to identify and to model the echo path with a high degree of accuracy.

It is supposed now that a_k and u_k are zero-mean stationary, mutually independent stochastic processes. u_k may have arbitrary statistics, whereas a_k is supposed as uncorrelated (i.e. white) data sequence with integer values. A generalization to data sequences with correlations (e.g. partial response signaling) is given in [1]. The variance of the transmitted data is denoted as $\sigma_a^2 = \mathrm{E}(a_k^2)$ and $\sigma_u^2 = \mathrm{E}(u_k^2)$ accordingly (E denotes expectation). The echo path is characterized by the impulse response vector of length L (T denotes transposition)

$$\boldsymbol{g} = (g_0, \ldots, g_{L-1})^{\mathrm{T}} \tag{1}$$

and the echo canceller is realized by an FIR filter with coefficient vector at time k

$$\boldsymbol{c}_k = (c_{k,0}, \ldots, c_{k,L-1})^{\mathrm{T}}. \tag{2}$$

For the data vector

$$\boldsymbol{a}_k = (a_k, \ldots, a_{k-L+1})^{\mathrm{T}} \tag{3}$$

applies $\mathrm{E}(\|\boldsymbol{a}_k\|^2) = \mathrm{E}(\boldsymbol{a}_k^{\mathrm{T}} \boldsymbol{a}_k) = L\sigma_a^2$. The output signals of echo path and echo canceller are given by scalar products:

$$y_k = \boldsymbol{a}_k^{\mathrm{T}} \boldsymbol{g}, \quad \hat{y}_k = \boldsymbol{a}_k^{\mathrm{T}} \boldsymbol{c}_k. \tag{4}$$

The residual (echo) signal φ_k can be written as follows using the coefficient misalignment vector $\boldsymbol{\varepsilon}_k$:

$$\varphi_k = y_k - \hat{y}_k = \boldsymbol{a}_k^{\mathrm{T}} \boldsymbol{\varepsilon}_k; \quad \boldsymbol{\varepsilon}_k := \boldsymbol{g} - \boldsymbol{c}_k. \tag{5}$$

3. LMS adaptation model

3.1 Infinite-precision algorithm (C-LMS)

The LMS algorithm is well-known for infinite-precision (continuous) operation [3, 4, 5] and therefore can be described shortly. The echo canceller adaptation is performed with the signal $u_k + \varphi_k$ occuring after the echo subtraction point

$$\boldsymbol{c}_{k+1} = \boldsymbol{c}_k + \alpha \boldsymbol{a}_k (u_k + \varphi_k). \tag{6}$$

The step size α is the only adjustable parameter to control the algorithm. **Fig. 2** shows the structure of the adaptive filter. If the coefficients $\boldsymbol{c}_k$ are set to their optimal values, then the vector of gradient estimation $\boldsymbol{a}_k(u_k + \varphi_k)$ is zero-mean. However, in every step the coefficients are altered again and hence a stabilization to the ideal settings is not possible. Even in steady-state, it remains a coefficient dithering, i.e. fluctuations around the optimal values. The amount of dithering is measured with the variance or mean squared value of the residual signal φ_k:

$$\sigma_\infty^2 = \lim_{k \to \infty} \sigma_k^2; \quad \sigma_k^2 = \mathrm{E}(\varphi_k^2). \tag{7}$$

The final steady-state error σ_∞^2 is defined as limit (if existing) of the residual signal variances. An important quantity is the ratio of residual signal to far-end signal

$$R_k^2 = \frac{\mathrm{E}(\varphi_k^2)}{\mathrm{E}(u_k^2)} = \frac{\sigma_k^2}{\sigma_u^2}. \tag{8}$$

In most applications

$$R_\infty^2 \overset{!}{\ll} 1 \tag{9}$$

is aimed, e.g. $R_\infty \approx -20 \ldots -40$ dB. At the same time, $+30$ dB is a typical value for the ratio of echo signal to far-end signal. Before the echo subtraction point the far-end signal is only a small "disturbance" of the echo, whereas after the subtraction point the echo should be merely a small disturbance of the far-end signal. As a consequence the ratio of residual signal to echo signal has to achieve about -60 dB.

The analysis of stochastic gradient algorithms becomes simpler by supposing the widely accepted so called *Independence-Theorem* introduced by Mazo [5]:

$$\boldsymbol{\varepsilon}_k \text{ und } \boldsymbol{a}_k \text{ are statistically independent.} \tag{10}$$

This implies $\mathrm{E}(\varphi_k) = 0$ and the residual signal variance results from the mean squared norm of the misalignment vector:

$$\sigma_k^2 = \mathrm{E}(\varphi_k^2) = \sigma_a^2\, \mathrm{E}(\|\boldsymbol{\varepsilon}_k\|^2). \tag{11}$$

3.2 Finite-precision algorithm (Q-LMS)

Every digital implementation of adaptive filtering involves coefficients quantized with a quantization step size q_c. Hence, the adaptation scheme (6) has to be modified so that the gradient estimator $\boldsymbol{a}_k(u_k+\varphi_k)$ is also quantized with q_c. For simplification q_c and α are supposed as power of 2. Therefore a definition of the infinite-precision adaptive algorithm applies as follows (see Fig. 2):

$$\boldsymbol{c}_{k+1}=\boldsymbol{c}_k+\alpha\boldsymbol{a}_k Q(u_k+\varphi_k). \tag{12}$$

The quantization function Q is assumed to be a symmetric staircase function with arbitrary transition points s_n. The difference between the equidistant quantization levels is the so-called quantization step size q which is assumed to be also power of 2. Finally, $Q(s_n)=(Q(s_n+0)+Q(s_n-0))/2$ is supposed for ease of theory. There are two cases to be considered:

1. 0 is not a transition point:
   $Q(x)=nq$ for $s_n<x<s_{n+1}$.
   This implies $q_c=\alpha q$.
2. 0 is a transition point:
   $Q(x)=(n+0.5)q$ for $s_n<x<s_{n+1}$.
   This implies $q_c=\alpha q/2$. Sign adaptation (S-LMS) is included here with $Q=\mathrm{sign}$, $q=2$, $q_c=\alpha$.

Generally, the optimal coefficients are given by $\boldsymbol{c}_{\mathrm{opt}}=\boldsymbol{g}$ and according to this, the misalignment vector $\boldsymbol{\varepsilon}_k$ was defined in (5). With Q-LMS, nevertheless, the coefficients are quantized with q_c and therefore a suboptimal quantized coefficient vector could be defined. However, this is insignificant because even in steady-state the coefficient dithering is much greater than the quantization step size:

$$q_c \ll \sigma_\infty. \tag{13}$$

4. Analysis of finite-precision adaptation

4.1 Iteration of residual signal variances

From the update model (12) the Euclidean norm is formed to yield

$$\|\boldsymbol{\varepsilon}_{k+1}\|^2=\|\boldsymbol{\varepsilon}_k\|^2-2\alpha\varphi_k Q(u_k+\varphi_k)+\alpha^2\|\boldsymbol{a}_k\|^2 Q^2(u_k+\varphi_k). \tag{14}$$

Taking expected values and with the aid of $\sigma_k^2=\sigma_a^2\,\mathrm{E}(\|\boldsymbol{\varepsilon}_k\|^2)$, a deterministic iteration scheme can be derived:

$$\frac{\sigma_{k+1}^2}{\sigma_a^2}=\frac{\sigma_k^2}{\sigma_a^2}-2\alpha\underbrace{\mathrm{E}(\varphi_k Q(u_k+\varphi_k))}_{=:W_k}+\alpha^2\underbrace{\mathrm{E}(\|\boldsymbol{a}_k\|^2 Q^2(u_k+\varphi_k)).}_{=:S_k} \tag{15}$$

Now the main problem is to calculate the expected values W_k und S_k.

In [14, 15] it is shown that the distribution of the residual signal converges to a limit distribution as $k\to\infty$, but only pure existence theorems are given without explicit calculation of the limit distribution. For sufficient large k_0, hence, the distribution of φ_k can be considered as nearly constant for $k>k_0$. This holds especially true for the distribution of $\varphi_{\mathrm{N},k}=\varphi_k/\sigma_k$, i.e. with standardization to unit variance. In other words, the shape of the probability density function (PDF) of the residual signal is assumed not to change with time. Since the calculation of W_k and S_k depends only on the distribution of φ_k, it follows that the index k may be omitted at u_k (stationary), $\boldsymbol{a}_k$ (stationary) and $\varphi_{\mathrm{N},k}$. With $\varphi_k=\sigma_k\varphi_{\mathrm{N},k}$ the expressions hold:

$$\begin{aligned} W_k&=W(\sigma_k); & W(\sigma)&=\mathrm{E}(\sigma\varphi_{\mathrm{N}}Q(u+\sigma\varphi_{\mathrm{N}}))\\ S_k&=S(\sigma_k); & S(\sigma)&=\mathrm{E}(\|\boldsymbol{a}\|^2 Q^2(u+\sigma\varphi_{\mathrm{N}})). \end{aligned} \tag{16}$$

W and S are both functions not only defined for a single argument σ_k but for the whole positive real axis. Hence, the functions $W, S:[0,\infty)\to\mathscr{R}$ are independent from k or the actual coefficient settings. W_k and S_k are resulting as functional values for the argument σ_k. Particularly for C-LMS, W and S reduce to parabolic curves

$$\begin{aligned} W(\sigma)&=\sigma^2\\ S(\sigma)&=L\sigma_a^2(\sigma_u^2+\sigma^2) \end{aligned} \tag{17}$$

and for S-LMS applies

$$\begin{aligned} W(\sigma)&=\mathrm{E}(\sigma\varphi_{\mathrm{N}}\,\mathrm{sign}(u+\sigma\varphi_{\mathrm{N}}))\\ S(\sigma)&=L\sigma_a^2. \end{aligned} \tag{18}$$

From (15) follows that Q-LMS can be described with a deterministic iterative update of the residual signal variances:

$$\sigma_{k+1}^2=\sigma_k^2-2\alpha\sigma_a^2 W(\sigma_k)+\alpha^2\sigma_a^2 S(\sigma_k). \tag{19}$$

This important attempt originates basically from Claasen, Mecklenbräuker [6]. A more compact expression is given by the function T defined as follows, which is also independent from k ($v_k=\sigma_k^2$):

$$\begin{aligned} v_{k+1}&=T(v_k)\\ T(v)&=v-2\alpha\sigma_a W(\sqrt{v})+\alpha^2\sigma_a^2 S(\sqrt{v}). \end{aligned} \tag{20}$$

Iterated application of T yields $v_k=T^k(v_0)$. If the iteration scheme (20) converges, then the final steady-state error $v_\infty=\sigma_\infty^2$ satisfies $T(\sigma_\infty^2)=\sigma_\infty^2$, i.e. σ_∞^2 is a fixed point of T. In [1] it is derived that convergence of the variance iteration (20) and therefore convergence of Q-LMS (12) is guaranteed if the condition

$$\|T\|_I=\sup_{v\in I}\left|1-\alpha\sigma_a^2\frac{W'(\sqrt{v})}{\sqrt{v}}+\frac{\alpha^2\sigma_a^2}{2}\frac{S'(\sqrt{v})}{\sqrt{v}}\right| \overset{!}{<} 1 \tag{21}$$

holds for an interval I with $T(I)\subset I$. Even with weak assumptions, the condition (21) can be ensured if the step size is chosen small enough. The proof is quite extensive [1] and does not yield any explicit expression for the steady-state error or the quantity $\|T\|_I$. The convergence of the algorithm is characterized by $\|T\|_I$ as follows: The deviation of the residual signal variance at time k to the final steady-state error, with respect to the start difference, is bounded by the k-th power of $\|T\|_I$:

$$\frac{|\sigma_k^2-\sigma_\infty^2|}{|\sigma_0^2-\sigma_\infty^2|}\leq(\|T\|_I)^k. \qquad (22)$$

For C-LMS especially, the following applies

$$T(v)=(1-2\alpha\sigma_a^2+\alpha^2L\sigma_a^4)v+\alpha^2L\sigma_a^4\sigma_u^2$$
$$\approx(1-2\alpha\sigma_a^2)v+\alpha^2L\sigma_a^4\sigma_u^2$$

and this implies the well-known results $\sigma_\infty^2\approx\alpha L\sigma_a^2\sigma_u^2/2$ and $\|T\|_I\approx 1-2\alpha\sigma_a^2$.

4.2 *Influence of the residual signal distribution*

A practicable calculation of the final steady-state error and the speed of convergence requires some further assumptions on the distributions of the residual signal and the far-end signal, exceeding the preceding supposed existence of a limit distribution of φ_k.

The distribution or limit distribution of the residual signal is usually taken as Gaussian [6, 9–13]. This seems a reasonable assumption for larger filter lengths L, but there has never been a theoretical analysis supporting this assumption, even not for infinite-precision operation. Now it has been established [1, 2] even for $L=1$, that the limit PDF of the residual signal is a slightly shaped Gaussian PDF or a sampled slightly shaped Gaussian PDF. "Slightly shaped" means exactly that convergence to the Gaussian PDF appears as $\alpha\rightarrow 0$, i.e. a second limit process is to be considered.

"Sampled" Gaussian PDF has also to be explained: The echo replica $\hat{y}$ is quantized with the quantization step size q_c in the same way as the filter coefficients. If the echo y would be also discrete-valued, then the PDF of the residual signal would be composed of Dirac delta functions spaced in the order of q_c. Since the final steady-state error is not in the order of q_c according to (13), the PDF of the residual signal can be supposed approximately as a continuous function. In addition it will be seen that the main results are mostly influenced only by the low-order moments of the residual signal distribution.

The PDF's of φ, $\varphi_N=\varphi/\sigma_\infty$, u, u/σ_u are denoted as f_φ, f_{φ_N}, f_u, f_{u_N} respectively. For the calculation of the function $W(\sigma)$ it is useful to introduce the auxiliary function

$$h_\varphi(x)=\int_{-\infty}^{x} e f_{\varphi_N}(e)\,\mathrm{d}e \quad (\leq 0). \qquad (23)$$

With some analysis [1] follows

$$W(\sigma)=-q\sigma^2\sum_n\int_{-\infty}^{\infty} h_\varphi(u)\,f_u(u\sigma+s_n)\,\mathrm{d}u. \qquad (24)$$

$W(\sigma)$ is determined primarily by the values of the PDF f_u at the transition points s_n of the quantization function Q, because h_φ attains its absolute maximum at point 0, so that the values of f_u in the neighbourhoods of s_n are most important. Approximately

$$h_\varphi(x)=-\exp(-x^2/2)/\sqrt{2\pi}$$

could be set up, if the equation (24) is to be evaluated without the simpler methods outlined in the next subsection.

4.3 *Further developments if the far-end signal PDF allows a Taylor series expansion*

The expression (24) gives already a very efficient method for a practicable calculation of $W(\sigma)$, provided that the PDF f_u is available in analytical representation. The utilization of f_u simplifies substantially and a better understanding of the function W results, if f_u allows a Taylor series expansion in the neighbourhoods of the transition points s_n:

$$f_u(u+s_n)=\sum_{r=0}^{\infty}\frac{f_u^{(r)}(s_n)}{r!}\,u^r. \qquad (25)$$

However, for applying this concept a strong validity of (25) is not required [1]. In many cases, the Taylor expansion may be abandoned after $r=0$ or $r=1$ without appreciable loss of accuracy. The value $f_u''(s_n)$ should be taken into account only for small values $f_u(s_n)$. If the PDF f_u is not available in analytical form but only as a set of measurements forming a histogram, nevertheless, it is rather easy to calculate estimations for the values of $f_u(s_n)$, $f_u'(s_n)$, $f_u''(s_n)$.

From (24) with the aid of (25), a result for $W(\sigma)$ can be derived and similar methods are applied to $S(\sigma)$ (notice $R=\sigma/\sigma_u$):

$$W(\sigma)=q\sigma_uR^2\sum_{r=0}^{\infty}R^{2r}M_{2r}\sum_n f_{u_N}^{(2r)}\left(\frac{s_n}{\sigma_u}\right) \qquad (26)$$

$$S(\sigma)=S(0)-2qL\sigma_a^2\sum_{r=0}^{\infty}R^{2r+2}\frac{M_{2r}}{2r+2}\cdot\sum_n Q(s_n)\,f_{u_N}^{(2r+1)}\left(\frac{s_n}{\sigma_u}\right) \qquad (27)$$

$$\text{with}\quad M_{2r}=\frac{\mathrm{E}(\varphi_N^{2r+2})}{(2r+1)!}\overset{(\times)}{=}\frac{1}{2^r r!} \qquad (28)$$

where $(\times)$ holds if the residual signal is exactly Gaussian. Direct from (16) is $S(0)=\mathrm{E}(\|a\|^2Q^2(u))=L\sigma_a^2\,\mathrm{E}(Q^2(u))$ evaluated. Obviously, $W(\sigma)$ depends only on the even-order and $S(\sigma)$ only on the odd-order derivatives of the PDF f_u at the transition points s_n. The residual signal distribution contributes only with its moments. The quantities M_{2r} converge fast to 0 and taking into account that $R\ll 1$ holds for the interesting scope of steady-state, a stop at small r in the series (26) and (27) seems reasonable.

In many applications the simplest case applies that the PDF f_u can be approximated with linear functions in the neighbourhoods of s_n. In the same way as for C-LMS, the functions $W(\sigma)$ and $S(\sigma)$ reduce to polynomials of degree 2 and $T(v)$ to a polynomial of degree 1 (i.e. linear function). A simple calculation yields

$$\sigma_\infty^2\approx\frac{\alpha L\sigma_a^2}{2W^*+\alpha S^*}\,\mathrm{E}(Q^2(u))$$
$$\|T\|_I\approx 1-2\alpha\sigma_a^2W^* \qquad (29)$$

where $W^*=q\sum_n f_u(s_n)$. S^* is less important for small α.

As a consequence, the final steady-state error becomes smaller and the speed of convergence becomes higher, if the values $f_u(s_n)$ become greater. For comparision, it is mentioned that with C-LMS the distribution of the far-end signal has no influence at all. A reduction of the step size α by a factor of 2 (i.e. increasing the coefficient wordlength by 1 bit) causes a decrease of the final steady-state error by 3 dB with both finite- and infinite-precision operation.

4.4 *Optimization of the quantization function*

Optimization of the quantization function means searching for that distribution of transition points s_n, which implies the most coarse quantization step size q_c of the coefficients with respect to the steady-state error beeing kept constant. If the far-end signal is approximately Gaussian distributed, the wanted optimum occurs with S-LMS. Thus, the adaptation scheme with the simplest implementation results at the same time in the smallest steady-state error, but unfortunately the speed of convergence is considerably reduced. If the PDF f_u has several distinct maxima, then S-LMS works poorer as Q-LMS with several transition points tuned to f_u.

An adaptive scheme with 3 transition points is the dual sign algorithm (DSA) [16] with favourable properties concerning both steady-state error and speed of convergence. The DSA is easy to analyse with the methods outlined above, because the transition points s_n can be chosen without any restrictions.

4.5 *Sign adaptation (S-LMS)*

Using only the signs of the gradient estimations $a_k(u_k+\varphi_k)$ for coefficient updating results in that adaptive scheme with the simplest implementation. **Fig. 3** shows the final steady-state error as a function of the step size α for several analytically defined distributions of the far-end signal (notice $\alpha_u=\alpha/\sigma_u=q_c/\sigma_u$, $R_\infty=\sigma_\infty/\sigma_u$).

The differences between (a) Gaussian, (b) uniform and (c) triangular distribution are negligible and hence in these cases an accurate knowledge of the value $f_{u_N}(0)$ is not required. But for the bimodal triangular distribution (i.e. the PDF consists of 2 triangles side by side) holds $f_{u_N}(0)=0$ and the gain in the steady-state error reduces to 2 dB/bit. For a pure binary far-end signal (e), the auxiliary function has been calculated with supposed Gaussian residual signal. However, for a so clear discrete far-end signal this assumption is not admissible and pretends better results than reality [2]. A binary far-end signal is the worst case for S-LMS. On the opposite, the best case occurs for the single-point distribution (i.e. the signal is constant zero and the PDF is the delta function) as limit case of a vanishing far-end signal (f). Of course, the algorithm works best if no superimposed far-end signal is disturbing. For comparision, Fig. 3 shows also the steady-state error for a coefficient setting fixed to the optimal quantized values (g).

The importance of the distribution of the far-end signal at the transition points s_n is recognized only in [6, 7, 9, 11]. Otherwise the far-end signal is simply assumed to be Gaussian, what can cause extreme errors depending on the real circumstances [2]. Modelling of Q by means of independent additive noise is also incorrect, if only because then the different curves in Fig. 3 become identical.

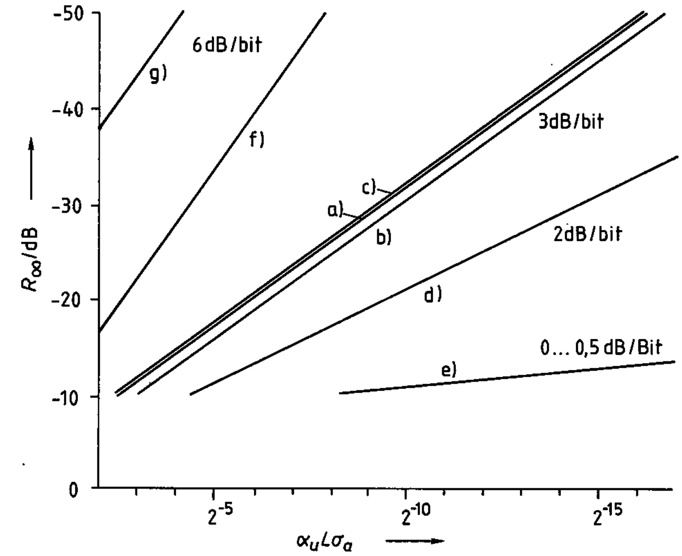

Fig. 3: Final steady-state error with S-LMS for several distributions of the far-end signal

a) Gaussian
b) Uniform
c) Triangular
d) Bimodal triangular
e) Binary (two-point distribution)
f) All-zero (single-point distribution)
g) Fixed optimal quantized coefficients

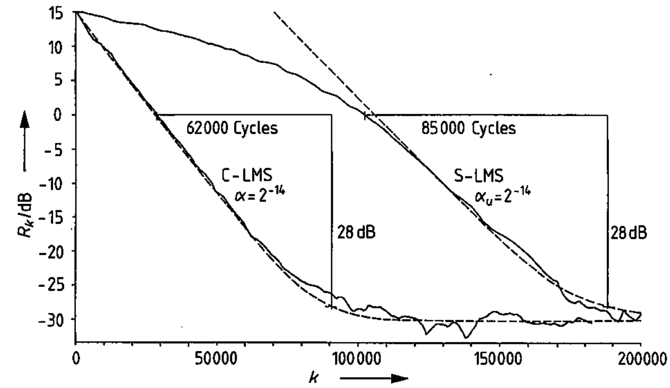

Fig. 4: Convergence with C- and S-LMS

5. Simulation results

The curves for $R_k=\sigma_k/\sigma_u$ in Figs. 4, 5, 6 are depicted as functions of $\alpha\sigma_a^2$ (C-LMS) resp. $\alpha_u\sigma_a^2=\alpha\sigma_a^2/\sigma_u$ (Q-LMS). The simulations are performed with $L=32$ coefficients and binary data a_k. Generally, simulation results are displayed with solid lines and theoretical results with dotted lines. The theoretical curves are calculated with

$$\sigma_k^2=\sigma_\infty^2+(\|T\|_I)^k\,(\sigma_0^2-\sigma_\infty^2)$$

according to (22).

C- and S-LMS are compared in **Fig. 4** for Gaussian u. To achieve the same steady-state error, the step size α for C-LMS must equal approximately $\alpha_u=\alpha/\sigma_u$ for S-LMS. The time to reach the converged steady-state is around 35% higher for S-LMS than for C-LMS, compared on the basis of identical steady-state errors. For C-LMS theoretical and simulation results agree very well. However, S-LMS indicates slower initial convergence (i.e. with

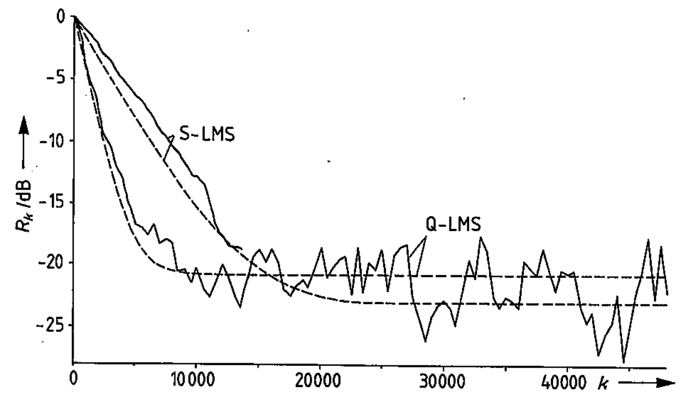

Fig. 5: Comparison of Q- and S-LMS for unimodal distributed far-end signal

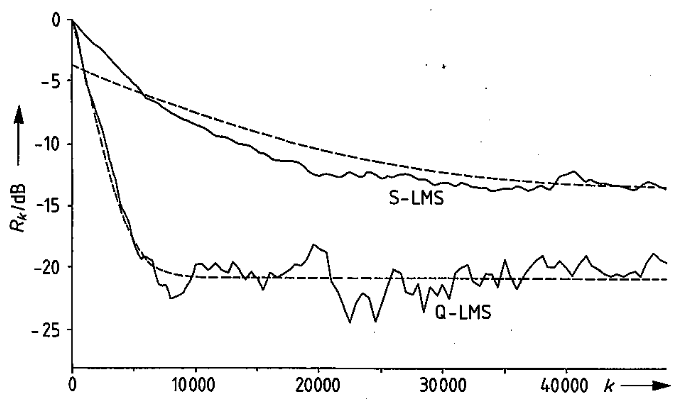

Fig. 6: Comparision of Q- and S-LMS for bimodal distributed far-end signal

$R_k > 0$ dB) than predicted by the theoretical curve. The reason is not hard to unterstand: The PDF of the far-end signal is approximated by means of a constant function, i.e, with its value for point 0. For $R_k > 1$ this approximation is still effective for points where the PDF does not match the constant (horizontal) approximation. However, the initial adaptation up to $R_k \approx 1$ is no serious problem if the step size is switched from large to small values. With an appropriate remove of the theoretical curve, a correct agreement with the simulation curve can be achieved even for S-LMS.

Now Q- and S-LMS are compared for two extremely different distributed far-end signals. In **Fig. 5** unimodal distribution (similar to Gaussian) and in **Fig. 6** bimodal distribution (with 2 distinct maxima and $f_u(0) = 0$) are assumed. The step size of 2^{-12} is identical for these 4 curves and Q-LMS is performed with $q' = \sigma_u$.

The comparision of the both S-LMS's shows the reduced speed of convergence and the increased steady-state error for the bimodal distribution compared with the unimodal distribution. To attain the same steady-state error with S-LMS would require for the bimodal distribution a step size smaller than a factor of 2^{-6} than that required for the unimodal distribution. This implies an extremely increasing difference in the speed of convergence.

The Figs. 5, 6 indicate clearly the differences when changing from S- to Q-LMS. For the unimodal distribution, Q-LMS is preferable in the speed of convergence but poorer in the final steady-state error compared with the bimodal distribution. For the bimodal distributed far-end signal, the adaptation algorithm works considerably better with additional transition points, i.e. with Q- than S-LMS.

6. Conclusion

The theory of finite-precision adaptive filtering presented in this contribution is based upon some fundamental methods: The adaptation scheme is expressed by means of a single quantization function and this implies every occuring signal to be quantized correctly without the need of additional nonlinear operations. Since the distribution shape of the residual signal does not change with time, the adaptation scheme is related with a 1-dimensional deterministic iteration model which can be investigated with the mathematical theorems for fixed points. The concept of Taylor series expansion for the PDF of the far-end signal allows simplified representations for the final steady-state error and for a quantity describing the speed of convergence which is both theoretically as practically important.

Summary of the main results: The amount of the remaining steady-state error and the speed of convergence depend on the shape of the PDF of the far-end signal, whereas in case of infinite-precision operation, only the power of the far-end signal is influential. Apart from some especially disadvantageous distributions which are simply to recognize, the sign algorithm is the adaptation scheme with the best results together with the simplest implementation. The formulas for the calculation of final steady-state error and speed of convergence are easy to apply and require only coarse knowledge of the distribution of the far-end signal.

References:

[1] Friedrichs, B.: Ein Beitrag zur Theorie und Anwendung wertdiskreter Adaptionsverfahren in digitalen Empfängern. Doctoral thesis Universität Erlangen-Nürnberg, 1990.
[2] Friedrichs, B.: Analysis of finite-precision adaptive filters – Part II: Computation of the residual signal distribution. To be published in Frequenz Vol. 46 (1992) 11—12.
[3] Widrow, B.; Stearns, S. D.: Adaptive signal processing. Englewood Cliffs: Prentice-Hall, 1985.
[4] Haykin, S.: Adaptive filter theory. Englewood Cliffs: Prentice-Hall, 1986.
[5] Mazo, J. E.: On the independence theory of equalizer convergence. Bell Syst. Tech. J. Vol. 58 (1979) pp. 963—993.
[6] Claasen, T. A. C. M.; Mecklenbräuker, W. F. G.: Comparison of the convergence of two algorithms for adaptive FIR digital filters. IEEE Trans. Acoust. Speech Signal Process. Vol. 29 (1981) 3, pp. 670—678.
[7] Verhoeckx, N. A. M.; Claasen, T. A. C. M.: Some considerations on the design of adaptive digital filters equipped with the sign algorithm. IEEE Trans. Commun. Vol. 32 (1984) 3, pp. 258—266.
[8] Verhoeckx, N. A. M.; van den Elzen, H. C.; Snijders, F. A. M.; van Gerwen, P. J.: Digital echo cancellation for baseband data transmission. IEEE Trans. Acoust. Speech Signal Process. Vol. 27 (1979) 6, pp. 768—781.
[9] Holte, N.; Stueflotten, S.: A new echo canceller for two-wire subscriber lines. IEEE Trans. Commun. Vol. 29 (1981) 11, pp. 1573—1581.
[10] Duttweiler, D. L.: Adaptive filter performance with nonlinearities in the correlation multiplier. IEEE Trans. Acoust. Speech Signal Process. Vol. 30 (1982) 4, pp. 578—586.
[11] Jørgensen, E.; Kjølaas, K. O.: Echo cancelling system based on the sign correlation algorithm. IEEE Global Telecommun. Conf. (GLOBECOM), 1981, C7.5.1-6.
[12] Kwong, C. P.: Dual sign algorithm for adaptive filtering. IEEE Trans. Commun. Vol. 34 (1986) 12, pp. 1272—1275.
[13] Sari, H.: Performance evaluation of three adaptive equalization algorithms. Proc. IEEE Int. Conf. Acoust. Speech Signal Process. (ICASSP), 1982, pp. 1385—1389.
[14] Bitmead, R. R.: Convergence in distribution of LMS-type adaptive parameter estimates. IEEE Trans. Autom. Control, Vol. 28 (1983) 1, pp. 54—60.
[15] Farden, D. C.: Stochastic approximation with correlated data. IEEE Trans. Inform. Theory, Vol. 27 (1981) 1, pp. 105—113.
[16] Mathews, V. J.: Performance analysis of adaptive filters equipped with the dual sign algorithm. IEEE Trans. Signal Process. Vol. 39 (1991) 1, pp. 85—91.

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